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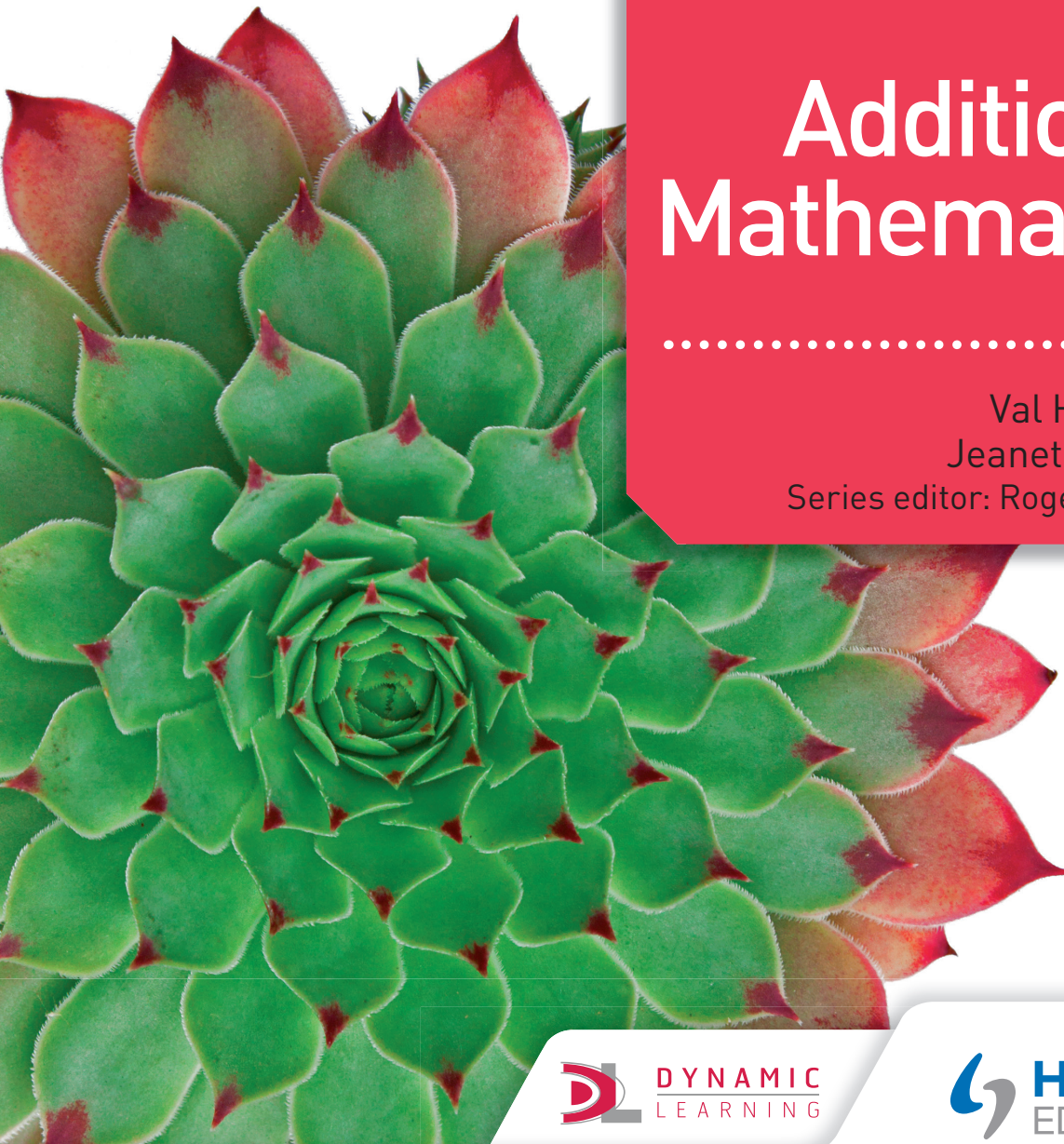
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Contents

3 Quadratic functions

Prior knowledge

You should be competent in the following skills:

★ factorising an expression of the form $ax^2 + bx + c$ where $a \neq 0$
★ solving a quadratic equation by factorising and using the quadratic formula
★ plotting or sketching a quadratic graph and using it to find an approximate solution to the related quadratic equation
★ using a graph to solve a quadratic inequality of the form $ax^2 + bx + c \geq 0$ and similar forms
★ solving a linear equation and a quadratic equation simultaneously algebraically or by sketching their graphs.

In this chapter you will learn how to:

★ find the maximum or minimum value of a quadratic function by any method
★ use the maximum or minimum value to determine the range for a given domain
★ identify whether a quadratic equation has two real distinct roots, two equal roots or no real roots
★ work with the equations of a given line and curve to identify whether the line will intersect the curve, be a tangent to the curve or not touch the curve
★ solve quadratic equations with real roots and find the solution set for quadratic inequalities.

Early mathematics focussed principally on arithmetic and geometry. However, in the sixteenth century a French mathematician, François Viète, started work on 'new algebra'. He was a lawyer by trade and served as a privy councillor to

▲ **Figure 3.1** François Viète (1540–1603)

both Henry III and Henry IV of France. His innovative use of letters and parameters in equations was an important step towards modern algebra.

Viète presented methods of solving equations of second, third and fourth degrees and discovered the connection between the positive roots of an equation and the coefficients of different powers of the unknown quantity. Another of Viète's remarkable achievements was to prove the fallacy of claims that a circle could be squared, an angle trisected and the cube doubled. He achieved all this, and much more, using only a ruler and compasses, without the use of either tables or a calculator!

💬 Discussion point

In order to appreciate the challenges he overcame, try to solve the quadratic equation $2x^2 - 8x + 5 = 0$ without using a calculator. Give your answers correct to 2 decimal places.

Maximum and minimum values

A **polynomial** is an expression where, apart from a constant term, the terms are positive integer powers of a variable. The highest power is the **order** of the polynomial. A **quadratic function** is a polynomial of order 2. For example $x^2 + 3$, a^2 and $2y^2 - 3y + 5$ are all examples of quadratic expressions. Each expression contains only one variable (letter), and the highest power of that variable is 2.

The graph of a quadratic function is either $\cup$-shaped or $\cap$-shaped. Think about the expression $x^2 + 3x + 2$. When the value of x is very large and positive or very large and negative, the x^2 term dominates resulting in large positive values. Consequently the graph of the function will be $\cup$-shaped.

Similarly, the term $-2x^2$ dominates the expression $5 - 4x - 2x^2$ for both large positive and negative values of x, giving negative values of the expression. Consequently the graph of this function is $\cap$-shaped.

While most of the quadratic equations that you will meet will have three terms, it is not uncommon to meet quadratic equations with only two terms. These fall into two main categories.

1 Equations with no constant term, for example $2x^2 - 5x = 0$. This has x as a common factor and so factorises to give
$$x(2x - 5) = 0$$
$$\Rightarrow x = 0 \text{ or } 2x - 5 = 0$$
$$\Rightarrow x = 0 \text{ or } x = 2.5$$

2 Equations with no 'middle' term, which come in two categories:

a The sign of the constant term is negative, for example $a^2 - 9 = 0$ giving
$$(a + 3)(a - 3) = 0$$
$$\Rightarrow a = -3 \text{ or } a = 3$$
This extends to examples such as $2a^2 - 7 = 0$
$$\Rightarrow a^2 = 3.5$$
$$\Rightarrow a = \pm\sqrt{3.5}$$

b The sign of the constant term is positive, for example $p^2 + 4 = 0$.
$$\Rightarrow p^2 = -4, \text{ so there is no real-valued solution.}$$

> ✏️ **Note**
>
> Depending on the calculator you are using, $\sqrt{(-4)}$ may be displayed as 'Math error' or '2i', where i is used to denote $\sqrt{(-1)}$. This is a **complex number** or **imaginary number** which you will meet if you study Further Mathematics at Advanced Level.

Graphs of all quadratic functions have a vertical line of symmetry. This gives a method of finding the maximum or minimum value. If the graph crosses the horizontal axis then the line of symmetry is halfway between the two points of intersection. The maximum or minimum value lies on this line of symmetry.
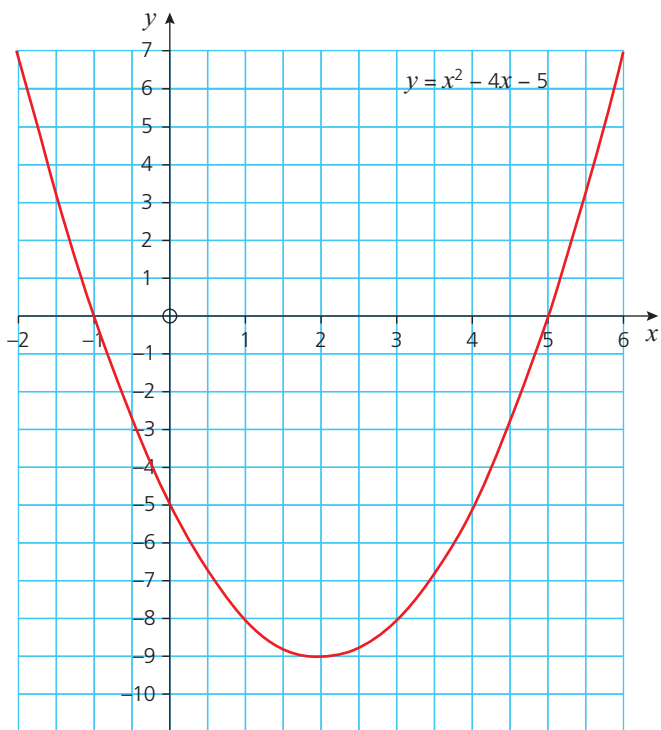
➡️ Worked example 3.1

1 Plot the curve $y = x^2 - 4x - 5$ for values of x from -2 to $+6$.

2 Identify the values of x where the curve intersects the horizontal axis.

3 Hence find the coordinates of the maximum or minimum point.

*This point is often referred to as the **turning point** of the curve.*

Solution

First create a table of values for $-2 \le x \le 6$

1

x	-2	-1	0	1	2	3	4	5	6
y	7	0	-5	-8	-9	-8	-5	0	7

This is also shown in the table.

Note

The line $x = 2$ passes through the turning point. It is a **vertical line of symmetry** for the curve.

2 The graph intersects the horizontal axis when $x = -1$ and when $x = 5$.

3 The graph shows that the curve has a minimum turning point halfway between $x = -1$ and $x = 5$. The table shows that the coordinates of this point are $(2, -9)$.

Drawing graphs by hand to find maximum or minimum values can be time-consuming. The following example shows you how to use algebra to find these values.

→ Worked example 3.2

Find the coordinates of the turning point of the curve $y = x^2 + x - 6$. State whether the turning point is a maximum or a minimum value.

Solution

The first step is to factorise the expression. One method of factorising is shown below, but if you are confident using a different method then continue to use it.

Find two integers (whole numbers) that multiply together to give the **constant** term, -6. Possible pairs of numbers with a product of -6 are:

6 and -1 1 and -6 3 and -2 2 and -3

There is only one pair, 3 and –2 whose sum is 1, the coefficient of x
So this is the pair that must be used to split up the $+x$ term.

$$x^2 + x - 6 = x^2 + 3x - 2x - 6$$
$$= x(x + 3) - 2(x + 3)$$
$$= (x + 3)(x - 2)$$

Both expressions in the brackets must be the same. Notice the sign change due to the minus sign in front of the 2.

> ### Note
>
> You would get the same result if you used $3x$ and $-2x$ in the opposite order:
> $$x^2 + x - 6 = x^2 - 2x + 3x - 6$$
> $$= x(x - 2) + 3(x - 2)$$
> $$= (x - 2)(x + 3)$$

The graph of $y = x^2 + x - 6$ will cross the x-axis when $(x + 3)(x - 2) = 0$, so that is when $x = -3$ and when $x = 2$.

The x coordinate of the turning point will be halfway between these two values, so:

$$x = \frac{-3 + 2}{2}$$
$$= -0.5$$

Substituting this value into the equation of the curve gives:

$$y = (-0.5)^2 + (-0.5) - 6$$
$$= -6.25$$

The equation of the curve has a positive x^2 term so will be $\cup$-shaped. It will therefore have a minimum value at $(-0.5, -6.25)$.

The method shown above can be adapted for curves where the coefficient of x^2 in the equation is not $+1$, for example $y = 6x^2 - 13x + 6$ and $y = 6 - x - 2x^2$.

➜ Worked example 3.3

For the curve with equation $y = 6 - x - 2x^2$:

1   Will the turning point of the curve be a maximum or a minimum? Give a reason for your answer.
2   Write down the coordinates of the turning point.
3   State the equation of the line of symmetry.

Solution

1 The coefficient of x^2 is negative so the curve will be $\cap$-shaped. This means the turning point will be a maximum.

2 First find two whole numbers that multiply together to give the product of the constant term and the coefficient of x^2, i.e. $6 \times (-2) = -12$.

   Possible pairs are:
   6 and -2; -6 and 2; 3 and -4; -3 and 4; 1 and -12; -1 and 12

Identify any of the pairs of numbers that can be added together to give the coefficient of x to be -1.

3 and -4 are the only pair with a sum of -1, so use this pair to split up the x term.

$$6 - x - 2x^2 = 6 + 3x - 4x - 2x^2$$
$$= 3(2 + x) - 2x(2 + x)$$
$$= (2 + x)(3 - 2x)$$

The graph of $y = 6 - x - 2x^2$ will cross the x-axis when $(2 + x)(3 - 2x) = 0$, so that is when $x = -2$ and when $x = 1.5$.

The x coordinate of the turning point is halfway between these two values:

$$x = \frac{-2 + 1.5}{2}$$
$$= -0.25$$

Substituting this value into the equation of the curve gives:

$$y = 6 - (-0.25) - 2(-0.25)^2$$
$$= 6.125$$

So the turning point is $(-0.25, 6.125)$.

3 The equation of the line of symmetry is $x = -0.25$.

Continue to use any alternative methods of factorising that you are confident with.

Both expressions in the brackets must be the same. Notice the sign change is due to the minus sign in front of the 2.

📝 Note

The methods shown in the above examples will always work for curves that cross the x-axis. For quadratic curves that do not cross the x-axis, you will need to use the method of **completing the square**.

➔ Worked example 3.4

1 Find the coordinates of the turning point of the quadratic function $f(x) = x^2 - 8x + 18$.

2 State whether it is a maximum or minimum.

3 Sketch the curve $y = f(x)$.

Solution

1 Start by halving the coefficient of x and squaring the result:

$-8 \div 2 = -4$

$(-4)^2 = 16$

Now use this result to break up the constant term, $+18$, into two parts:

$18 = 16 + 2$

and use this to rewrite the original expression as:

$f(x) = x^2 - 8x + 16 + 2$

$= (x - 4)^2 + 2$

$(x - 4)^2 \geq 0$ (always)

$\Rightarrow f(x) \geq 2$ for all values of x

You will always have a perfect square in this expression.

When $x = 4$, $f(x) = 2$
So the turning point is $(4, 2)$.

Substituting $x = 4$ in $x^2 - 8x + 18$

2 $f(x) \geq 2$ for all values of x so the turning point is a minimum.

3 The function is a $\cup$-shaped curve because the coefficient of x is positive. From the above, the minimum turning point is at $(4, 2)$, so the curve does not cross the x-axis. To sketch the graph, you will also need to know where it crosses the y-axis.

$f(x) = x^2 - 8x + 18$ crosses the y-axis when $x = 0$, i.e. at $(0, 18)$.
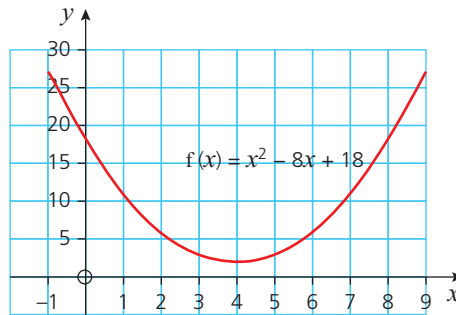
Note

Sometimes you will only need to sketch the graph of a function $f(x)$ for certain values of x. In this case, this set of values is referred to as the **domain** of the function, and the corresponding set of y values is called the **range**.

→ Worked example 3.5

Use the method of completing the square to work out the coordinates of the turning point of the quadratic function $f(x) = 2x^2 - 8x + 9$.

Solution

$$f(x) = 2x^2 - 8x + 9$$
$$= 2(x^2 - 4x) + 9$$
$$= 2((x - 2)^2 - 4) + 9$$
$$= 2(x - 2)^2 + 1$$

$(x - 2)^2 \geq 0$ (always), so the smallest value of $f(x)$ is 1.
When $f(x) = 1$, $x = 2$.
Therefore, the function $f(x) = 2x^2 - 8x + 9$ has a minimum value of $(2, 1)$.

→ Worked example 3.6

The domain of the function $y = 6x^2 + x - 2$ is $-3 \leq x \leq 3$.
Sketch the graph and find the range of the function.

Solution

The coefficient of x^2 is positive, so the curve will be $\cup$-shaped and the turning point will be a minimum.
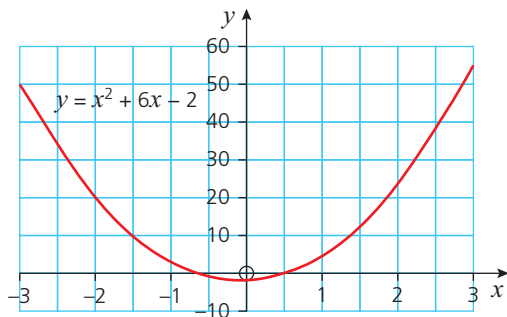The curve crosses the x-axis when $6x^2 + x - 2 = 0$.

$$6x^2 + x - 2 = (3x + 2)(2x - 1)$$
$$\Rightarrow (3x + 2)(2x - 1) = 0$$
$$\Rightarrow (3x + 2) = 0 \text{ or } (2x - 1) = 0$$

So the graph crosses the x-axis at $\left(-\frac{2}{3}, 0\right)$ and $\left(\frac{1}{2}, 0\right)$.

The curve crosses the y-axis when $x = 0$, so this is at $(0, -2)$.

$y = x^2 + 6x - 2$

The curve has a vertical line of symmetry passing halfway between the two points where the curve intersects the x-axis. Therefore, the equation of this line of symmetry is $x = \frac{-\frac{2}{3} + \frac{1}{2}}{2}$ or $x = -\frac{1}{12}$.

When $x = -\frac{1}{12}$, $y = 6\left(-\frac{1}{12}\right)^2 + \left(-\frac{1}{12}\right) - 2 = -2\frac{1}{24}$, the minimum value of the function.

To find the range, work out the values of y for $x = -3$ and $x = +3$.

When $x = -3$, $y = 6(-3)^2 + (-3) - 2 = 49$.

The larger of these will give the maximum value.

When $x = 3$, $y = 6(-3)^2 + 3 - 2 = 55$.

The range of the function corresponding to the domain $-3 \leq x \leq 3$

is therefore $-2\frac{1}{24} \leq y \leq 55$.

Exercise 3.1

1 Solve each equation by factorising.
   a $x^2 + x - 20 = 0$ **b** $x^2 - 5x + 6 = 0$
   c $x^2 - 3x - 28 = 0$ **d** $x^2 + 13x + 42 = 0$

2 Solve each equation by factorising.
   a $2x^2 - 3x + 1 = 0$ **b** $9x^2 + 3x - 2 = 0$
   c $2x^2 - 5x - 7 = 0$ **d** $3x^2 + 17x + 10 = 0$

3 Solve each equation by factorising.
   a $x^2 - 169 = 0$ **b** $4x^2 - 121 = 0$
   c $100 - 64x^2 = 0$ **d** $12x^2 - 27 = 0$

4 For each of the following curves:
   i) Factorise the function.
   ii) Work out the coordinates of the turning point.
   iii) State whether the turning point is a maximum or minimum.
   iv) Sketch the graph, labelling the coordinates of the turning point and any points of intersection with the axes.
   a $y = x^2 + 7x + 0$ **b** $y = 5 - 9x - 2x^2$
   c $f(x) = 16 - 6x - x^2$ **d** $f(x) = 2x^2 + 11x + 12$

5 Rewrite these quadratic expressions in the form $(x + a)^2 + b$.
   a $x^2 + 4x + 9$ **b** $x^2 - 10x - 4$
   c $x^2 + 5x - 7$ **d** $x^2 - 9x - 2$

6 Rewrite these quadratic expressions in the form $c(x + a)^2 + b$.
   a $2x^2 - 12x + 5$ **b** $3x^2 + 12x + 20$
   c $4x^2 - 8x + 5$ **d** $2x^2 + 9x + 6$

7 Solve the following quadratic equations. Leave your answers in the form $x = p \pm \sqrt{q}$.
   a $x^2 + 4x - 9 = 0$ **b** $x^2 - 7x - 2 = 0$
   c $2x^2 + 6x - 9 = 0$ **d** $3x^2 + 9x - 15 = 0$

8 For each of the following functions:
   i) Use the method of completing the square to find the coordinates of the turning point of the associated curve.
   ii) State whether the turning point is a maximum or a minimum.
   iii) Sketch the curve.
   a $f(x) = x^2 + 6x + 15$ **b** $y = 8 + 2x - x^2$
   c $y = 2x^2 + 2x - 9$ **d** $f: x \to x^2 - 8x + 20$

9 Sketch the graph and find the corresponding range for each function and domain.
   a $y = x^2 - 7x + 10$ for the domain $1 \le x \le 6$
   b $f(x) = 2x^2 - x - 6$ for the domain $-2 \le x \le 2$

Real-world activity

a Draw a sketch of a bridge modelled on the equation $25y = 100 - x^2$ for $-10 \le x \le 10$. Label the origin O, the point A(−10, 0), the point B(10, 0) and the point C(0, 4).
b 1 unit on your graph represents 1 metre. State the maximum height of the bridge, OC, and the span, AB.
c Work out the equation of a similar bridge with a maximum height of 5 m and a span of 40 m.

The quadratic formula

The **roots** of a quadratic equation $f(x)$ are those values of x for which $y = 0$ for the curve $y = f(x)$. In other words, they are the x coordinates of the points where the curve either crosses or touches the x-axis. There are three possible outcomes:

» The curve crosses the x-axis at two distinct points. In this case, the corresponding equation is said to have *two real distinct roots*.
» The curve touches the x-axis, in which case the equation has *two equal (repeating) roots*.
» The curve lies completely above or completely below the x-axis so it neither crosses nor touches the axis. In this case the equation has *no real roots*.

The next example uses the method of completing the square introduced earlier, showing how to generalise it to give a formula for solving quadratic equations.

→ **Worked example 3.7**

Solve $2x^2 + x - 4 = 0$.

Solution   ## Generalisation

$2x^2 + x - 4 = 0$

$\Rightarrow x^2 + \frac{1}{2}x - 2 = 0$

$\Rightarrow x^2 + \frac{1}{2}x = 2$

$\Rightarrow x^2 + \frac{1}{2}x + \left(\frac{1}{4}\right)^2 = 2 + \left(\frac{1}{4}\right)^2$

$\Rightarrow \left(x + \frac{1}{4}\right)^2 = \frac{33}{16}$

$\Rightarrow \left(x + \frac{1}{4}\right) = \pm\frac{\sqrt{33}}{4}$

$\Rightarrow x = -\frac{1}{4} \pm \frac{\sqrt{33}}{4}$

$\qquad = \frac{-1 \pm \sqrt{33}}{4}$

$ax^2 + bx + c = 0$

$\Rightarrow x^2 + \frac{b}{a}x + \frac{c}{a} = 0$

$\Rightarrow x^2 + \frac{b}{a}x = -\frac{c}{a}$

$\Rightarrow x^2 + \left(\frac{b}{a}\right)x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$

$\Rightarrow \left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$

$\qquad\qquad = \frac{b^2 - 4ac}{4a^2}$

$\Rightarrow x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$

$\qquad\qquad = \pm\frac{\pm\sqrt{b^2 - 4ac}}{2a}$

$\Rightarrow x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$

$\qquad = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

The result $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ is known as the **quadratic formula**. You can use it to solve any quadratic equation. One root is found by taking the + sign, and the other by taking the – sign. When the value of $b^2 - 4ac$ is negative, the square root cannot be found and so there is no real solution to that quadratic equation. This occurs when the curve does not cross the x-axis.

✎ **Note**

The part $b^2 - 4ac$ is called the **discriminant** because it discriminates between equations with roots and those with no roots. It also tells you when a quadratic equation has two real roots that are distinct or two equal roots.

In an equation of the form $(px + q)^2$, where p and q can represent either positive or negative numbers, $px + q = 0$ gives the only solution.

The quadratic formula implies that a quadratic equation will only have a repeated root if the discriminant is zero.

➡ Worked example 3.8

Show that the equation $4x^2 - 12x + 9 = 0$ has a repeated root by:
1 factorising and
2 using the discriminant.

Solution

1 $\qquad 4x^2 - 12x + 9 = 0$
$\Rightarrow (2x - 3)(2x - 3) = 0$
$\Rightarrow 2x - 3) = 0$
$\Rightarrow x = 1.5$
2 The discriminant $b^2 - 4ac = (-12)^2 - 4(4)(9) = 0$

In some cases, such as in the previous example, the factorisation is not straightforward. In such cases, evaluating the discriminant is a reliable method to obtain an accurate result.

➡ Worked example 3.9

Show that the equation $3x^2 - 2x + 4 = 0$ has no real solution.

Solution

The most straightforward method is to consider the discriminant. If the discriminant is negative, there is no real solution.
Comparing $3x^2 - 2x + 4 = 0$ with $ax^2 + bx + c$ gives $a = 3$, $b = -2$ and $c = 4$.
Substituting these values into the discriminant:
$\qquad b^2 - 2ac = (-2)^2 - 4 \times 3 \times 4$
$\qquad\qquad = -44$
Since the discriminant is negative, there is no real solution.

So far the examples have considered whether or not a curve intersects, touches or lies completely above or below the x-axis. The next example considers the more general case of whether or not a curve intersects, touches or lies completely above or below a particular straight line. Remember that the x-axis has equation $y = 0$ and the general form of the equation of a straight line is $y = mx + c$. (This has alternate forms, e.g. $ax + by + c = 0$.)

➡ Worked example 3.10

1 Find the coordinates of the points of intersection of the line $y = 4 - 2x$ and the curve $y = x^2 + x$.

2 Sketch the line and the curve on the same axes.

Solution

1 To find where a curve $y = f(x)$ intersects the x-axis ($y = 0$), we solved the curve equation simultaneously with the equation of the x-axis, $y = 0$.

Here, we solve $y = x^2 + x$ simultaneously with $y = 4 - 2x$. At the points of intersection, the y values of both equations will be the same.

$$x^2 + x = 4 - 2x$$
$$\Rightarrow x^2 + 3x - 4 = 0$$
$$\Rightarrow (x + 4)(x - 1) = 0$$
$$\Rightarrow x = -4 \text{ or } x = 1$$

It is more straightforward to substitute into the line equation.

To find the y coordinate, substitute into one of the equations.

When $x = -4$, $y = 4 - 2(-4) = 12$.

When $x = 1$, $y = 4 - 2(1) = 2$.

The line $y = 4 - 2x$ intersects the curve $y = x^2 + x$ at $(-4, 12)$ and $(1, 2)$.

2 The curve has a positive coefficient of x^2 so is $\cup$-shaped.

It crosses the x-axis when $x^2 + x = 0$.

$$\Rightarrow x(x + 1) = 0$$
$$\Rightarrow x = 0 \text{ or } x = -1$$

So the curve crosses the x-axis at $x = 0$ and $x = -1$.

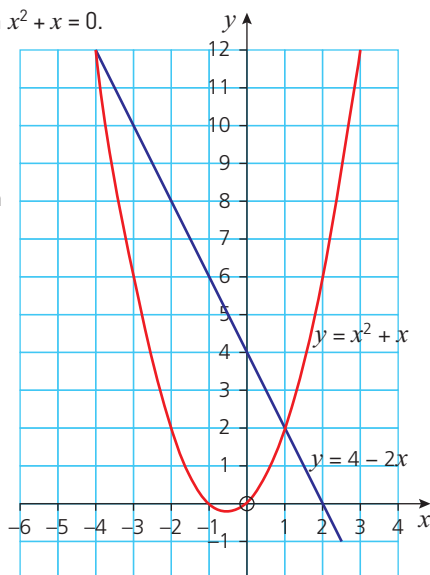It crosses the y-axis when $x = 0$.

Substituting $x = 0$ into $y = x^2 + x$ gives $y = 0$.

So the curve passes through the origin.

The line $2x + y = 4$ crosses the x-axis when $y = 0$. When $y = 0$, $x = 2$.

The line $2x + y = 4$ crosses the y-axis when $x = 0$, $y = 4$.

In the same way as a curve may touch the x-axis, it is possible for a quadratic curve to touch a general line, either sloping or parallel to the x-axis. Again, the equations of the line and curve need to be solved simultaneously, but in this case you will have a repeated value of x, implying that they meet at (touch) only one point.

➡ Worked example 3.11

1 Use algebra to show that the line $y = 6x - 19$ touches the curve $y = x^2 - 2x - 3$ and find the coordinates of the point of contact.

2 Sketch the line and curve on the same axes.

Solution

1 Solve the equations simultaneously by substituting the expression for y from one equation into the other:

$$x^2 - 2x - 3 = 6x - 19$$
$$\Rightarrow x^2 - 8x + 16 = 0$$
$$\Rightarrow (x - 4)^2 = 0$$
$$\Rightarrow x = 4$$

The repeated root $x = 4$ shows that the line and the curve touch. Substitute $x = 4$ into either equation to find the value of the y coordinate:

It is more straightforward to substitute into the line equation.

$$y = 6x - 19$$
$$\Rightarrow y = 6(4) - 19$$
$$= 5$$

Therefore the point of contact is (4, 5).

2 The coefficient of x^2 is positive so the curve will be $\cup$-shaped.

Substituting $x = 0$ into $y = x^2 - 2x - 3$ shows that the curve intersects the y-axis at (0, –3).

Substituting $y = 0$ into $y = x^2 - 2x - 3$ gives $x^2 - 2x - 3 = 0$.

$$\Rightarrow (x - 3)(x + 1) = 0$$
$$\Rightarrow x = -1 \text{ or } x = 3$$

So the curve intersects the x-axis at (–1, 0) and (3, 0).

The tangent, $y = 6x - 19$, touches the curve at (4, 5), intersects the y-axis at (0, –19) and the x-axis at $\left(3\frac{1}{6}, 0\right)$.
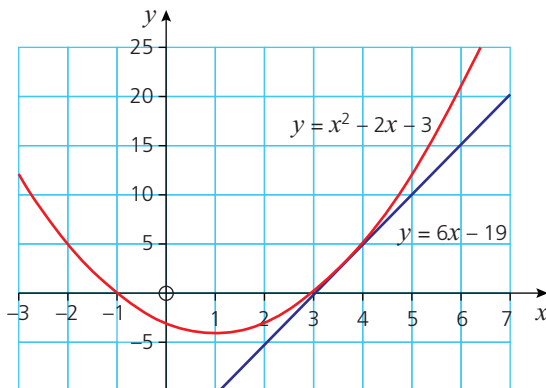
There are many situations when a line and a curve do not intersect or touch each other. A straightforward example of this occurs when the graph of a quadratic function is a ∪-shaped curve completely above the x-axis, e.g. $y = x^2 + 3$, and the line is the x-axis.

You have seen how solving the equations of a curve and a line simultaneously will give a quadratic equation with *two* roots when the line crosses the curve, and a quadratic equation with a *repeated* root when it touches the curve. If solving the two equations simultaneously results in *no* real roots, i.e. the discriminant is negative, then they neither cross nor touch.

Discussion point

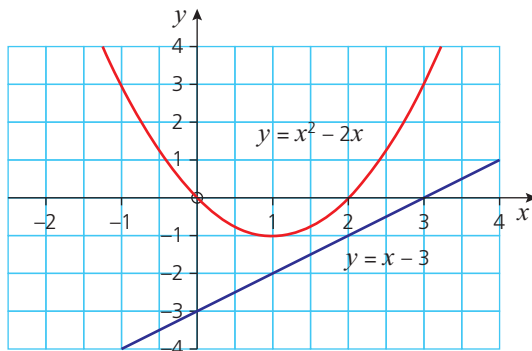Why is it not possible for a quadratic curve to touch a line parallel to the y-axis?

➡ Worked example 3.12

1   Sketch the graphs of the line $y = x - 3$ and the curve $y = x^2 - 2x$ on the same axes.
2   Use algebra to prove that the line and the curve don't meet.

Solution

1

2 Solving the two equations simultaneously:

$x^2 - 2x = x - 3$

$\Rightarrow x^2 - 3x + 3 = 0$

This does not factorise, so solve using the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$a = 1$, $b = 2$ and $c = 3$

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(3)}}{2(1)}$$

$$= \frac{3 \pm \sqrt{-3}}{2}$$

Since there is a negative value under the square root, there is no *real* solution. This implies that the line and the curve do not meet.


📝 Note

It would have been sufficient to consider only the discriminant $b^2 - 4ac$. Solving a quadratic equation is equivalent to finding the point(s) where the curve crosses the horizontal axis (the roots).

➡️ Worked example 3.13

A triangle has a base of $(2x + 1)$ cm, a height of x cm and an area of 68 cm^2.

1 Show that x satisfies the equation $2x^2 + x - 136 = 0$.

2 Solve the equation and work out the base length of the triangle.
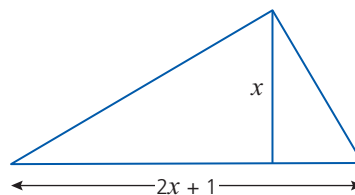
Solution

1 Using the formula for the area of a triangle, *area* $= \frac{1}{2}$ *base* × *height*:

Area $= \frac{1}{2} \times (2x + 1) \times x$

$= \frac{1}{2}(2x^2 + x)$

The area is 68 cm^2, so:

$\frac{1}{2}(2x^2 + x) = 68$

$\Rightarrow 2x^2 + x = 136$

$\Rightarrow 2x^2 + x - 136 = 0$

2 It is not easy to factorise this equation – it is not even obvious that there are factors – so using the quadratic formula:

Alternatively, you can use a graphics calculator.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$a = 2$, $b = 1$ and $c = -136$

$$x = \frac{-1 \pm \sqrt{1^2 - 4(2)(-136)}}{2(2)}$$

$$\Rightarrow x = \frac{-1 \pm \sqrt{1089}}{4}$$

$$\Rightarrow x = \frac{-1 \pm 33}{4}$$

$$\Rightarrow x = 8 \text{ or } x = -8.5$$

Since x is a length, reject the negative solution.

Substitute $x = 8$ into the expression for the base of the triangle, $(2x + 1)$, and work out the length of the base of the triangle, 17 cm.

✎ **Note**

Check that this works with the information given in the original question. $\frac{1}{2} \times 17 \text{ cm} \times 8 \text{ cm} = 68 \text{ cm}^2$

Solving quadratic inequalities

The quadratic inequalities in this section all involve quadratic expressions that factorise. This means that you can find a solution either by sketching the appropriate graph or by using line segments to reduce the quadratic inequality to two simultaneous linear inequalities.

The example below shows two valid methods for solving quadratic inequalities. You should use whichever method you prefer. Your choice may depend on how easily you sketch graphs or if you have a graphic calculator which will do this for you.

➡ **Worked example 3.14**

Solve these quadratic inequalities.

1 $x^2 - 2x - 3 < 0$
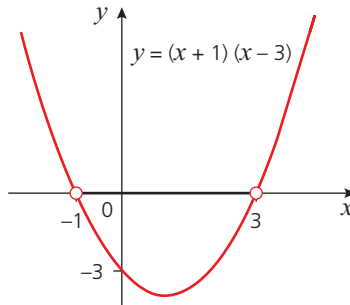
2 $x^2 - 2x - 3 \geq 0$

Solution

Method 1

$x^2 - 2x - 3 = (x + 1)(x - 3)$

So the graph of $y = x^2 - 2x - 3$ crosses the x-axis when $x = -1$ and $x = 3$.
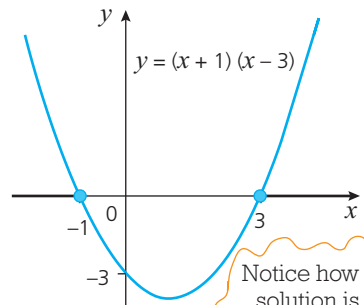
Look at the two graphs below:

1

Here the end points *are not* included in the solution, so you draw open circles: ○

$y = (x + 1)(x - 3)$

The solution is $-1 < x < 3$.

2

Here the end points *are* included in the solutions, so you draw solid circles: ●

$y = (x + 1)(x - 3)$

The solution is $x \leq -1$ or $x \geq 3$.

Notice how the solution is in two parts when there are two line segments.

1 The answer is the values of x for which $y < 0$, i.e. where the curve is below the x-axis.

2 The answer is the values of x for which $y \geq 0$, i.e. where the curve crosses or is above the x-axis.

Method 2

This method identifies the values of x for which each of the factors is 0 and considers the sign of each factor in the intervals between these critical values.

	$x < -1$	$x = -1$	$-1 < x < 3$	$x = 3$	$x > 3$
Sign of $(x + 1)$	–	0	+	+	+
Sign of $(x - 3)$	–	–	–	0	+
Sign of $(x + 1)$ $(x - 3)$	$(-) \times$ $(-) = +$	$(0) \times$ $(-) = 0$	$(+) \times (-)$ $= -$	$(+) \times$ $(0) = 0$	$(+) \times$ $(+) = +$

From the table, the solution to:

1 $(x + 1)(x - 3) < 0$ is $-1 < x < 3$

2 $(x + 1)(x - 3) \geq 0$ is $x \leq -1$ or $x \geq 3$

✎ Note

If the inequality to be solved contains $>$ or $<$, then the solution is described using $>$ and $<$, but if the original inequality contains $\geq$ or $\leq$, then the solution is described using $\geq$ and $\geq$.
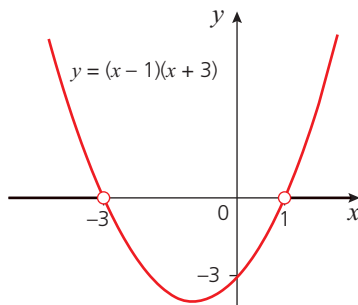
If the quadratic inequality has the variable on both sides, you must first collect everything on one side, as you would do when solving a quadratic equation.

➡ Worked example 3.15

Solve $2x + x^2 > 3$.

Solution

$2x + x^2 > 3 \Rightarrow x^2 + 2x - 3 > 0$

$\Rightarrow (x - 1)(x + 3) > 0$

From the graph, the solution is < -3 or $x > 1$.

Exercise 3.2

1 For each of the following equations, decide if there are two real and different roots, two equal roots or no real roots. Solve the equations with real roots.

a $x^2 + 3x + 2 = 0$

b $t^2 - 9 = 0$

c $x^2 + 16 = 0$

d $2x^2 - 5x = 0$

e $p^2 + 3p - 18 = 0$

f $x^2 + 10x + 25 = 0$

g $15a^2 + 2a - 1 = 0$

h $3r^2 + 8r = 3\,2$

2 Solve the following equations:

i) by completing the square

ii) by using the quadratic formula.

Give your answers correct to 2 decimal places.

a $x^2 - 2x - 10 = 0$

b $x^2 + x = 10$

c $2x^2 + 2x - 9 = 0$

d $2x^2 + x - 8 = 0$

3 Try to solve each of the following equations. Where there is a solution, give your answers correct to 2 decimal places.

a $4x^2 + 6x - 9 = 0$

b $9x^2 + 6x + 4 = 0$

c $(2x + 3)^2 = 7$

d $x(2x - 1) = 9$

4 Use the discriminant to decide whether each of the following equations has two equal roots, two distinct roots or no real roots.

a $9x^2 - 12x + 4 = 0$

b $6x^2 - 13x + 6 = 0$

c $2x^2 + 7x + 9 = 0$

d $2x^2 + 9x + 10 = 0$

e $3x^2 - 4x + 5 = 0$

f $4x^2 + 28x + 49 = 0$

5 Each pair of equations gives one curve and one line. For each pair, use any valid method to determine if the line intersects the curve, is a tangent to the curve or does not meet the curve. Give the coordinates of any points where the line and curve touch or intersect.

a $y = x^2 - 12x$; $y = 9 - 6x$

b $y = 2x^2 + 3x - 4$; $y = 2x - 6$

c $y = 6x^2 - 12x + 6$; $y = x$

d $y = x^2 - 8x + 18$; $y = 2x + 3$

e $y = x^2 + x$; $2x + y = 4$

f $y = 4x^2 + 9$; $y = 12x$

g $y = 3 - 2x - x^2$; $y = 9 + 2x$

h $y = (3 - 2x)^2$; $y = 2 - 3x$

6 Solve the following inequalities and illustrate each solution on a number line.

a $x^2 - 6x + 5 > 0$

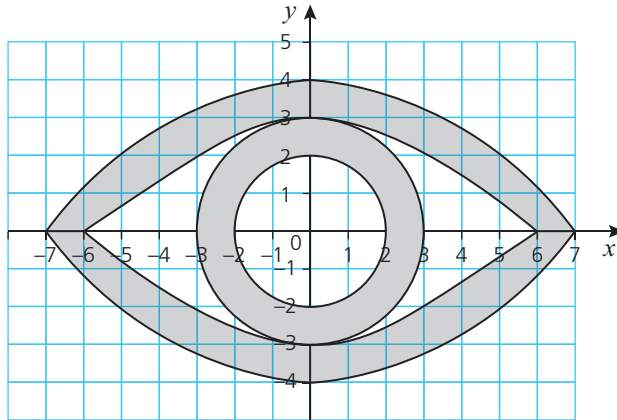**b** $a^2 + 3a - 4 \leq 0$

c $4 - y^2 > 0$

d $x^2 - 4x + 4 > 0$

e $8 - 2a > a^2$

f $3y^2 + 2y - 1 > 0$

> ### ▶ Real-world activity
>
> Anna would like to design a brooch for her mother and decides that it should resemble an eye. She starts by making the scale drawing, shown below.
>
>
>
> The brooch is made up of the shaded area.
>
> The equations of the two circles are $x^2 + y^2 = 4$ and $x^2 + y^2 = 9$.
>
> The rest of the brooch is formed by quadratic curves.
>
> The scale is 2 units represents 1 cm.
>
> **a** Find the equations of the four quadratic curves.
> **b** Anna then decides to design some earrings using a smaller version of the brooch design. She reduces the size by a factor of 2. Find the equations of the four quadratic curves for the earring design.

Key points ✔

✔ A **quadratic function** has the form $f(x) = ax^2 + bx + c$, where a, b and c can be any number (positive, negative or zero) provided that $a \neq 0$. The set of possible values of x is called the **domain** of the function and the set of y values is called the **range**.

✔ To plot the graph of a quadratic function, first calculate the value of y for each value of x in the given range.

✔ The graph of a quadratic function is symmetrical about a vertical line. It will be $\cup$-shaped if the coefficient of x^2 is positive and $\cap$-shaped if the coefficient of x^2 is negative.

✔ To sketch the graph of a quadratic function, look at the coefficient of x^2 to determine the shape, substitute $x = 0$ to determine where the curve crosses the vertical axis and solve $f(x) = 0$ to determine any values of x where the curve touches or crosses the horizontal axis. If there are no real values for x for which $f(x) = 0$, then the curve will be either completely above or completely below the x-axis.

✔ A **quadratic equation** is of the form $ax^2 + bx + c = 0$, with $a \neq 0$.

✔ To **factorise** a quadratic equation of the form $x^2 + bx + c = 0$, look for two numbers, p and q, with the sum b and the product c. The factorised form is then $(x - p)(x - q) = 0$. To factorise an equation of the form $ax^2 + bx + c = 0$, look for two numbers with the sum b and the product ac.

✔ The **discriminant** of a quadratic equation $ax^2 + bx + c = 0$ is $b^2 - 4ac$. If $b^2 - 4ac > 0$, a quadratic equation will have two distinct solutions (or roots). If $b^2 - 4ac = 0$, the two roots are equal so there is one repeating root. If $b^2 - 4ac < 0$, the roots have no real values.

✔ An expression of the form $(px + q)^2$ is called a **perfect square**.

✔ $x^2 + bx + c$ can be written as $\left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c$ using the method of **completing the square**. For expressions of the form $ax^2 + bx + c$, first take a out as a factor.

✔ The **quadratic formula** for solving an equation of the form $ax^2 + bx + c = 0$ is $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

✔ To find the point(s) where a line and a curve touch or intersect, substitute the expression for y from one equation into the other to give a quadratic equation in x.

✔ When solving a **quadratic inequality**, it is advisable to start by sketching the associated quadratic graph.

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